

Some results about the facial geometry of convex semi-infinite systems

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ABSTRACT. We study the geometrical properties of the convex semi-infinite systems and their solution sets. Our main focus is on those systems enjoying the so-called locally Farkas-Minkowski property. The paper provides convex counterparts of some results already proven for linear systems, pointing out the main differences, and finding sufficient conditions for their fulfilment.

1. Introduction

The paper deals with *convex inequality systems* in the form

$$\sigma = \{f_t(x) \leq 0, t \in T\},$$

where $f_t : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ is a lower semicontinuous (l.s.c., in brief) proper convex function, for $t \in T$, and T is an arbitrary (possibly infinite) index set. The *solution set* of σ is a (possibly empty) closed convex set F , and σ is *consistent* if F is non-empty.

If $f_t(x) = \langle a_t, x \rangle - b_t$, $t \in T$, with x and a_t in \mathbb{R}^n , b_t in \mathbb{R} , and $\langle \cdot, \cdot \rangle$ represents the inner product in \mathbb{R}^n , we obtain the *linear semi-infinite system*

$$\sigma = \{\langle a_t, x \rangle \leq b_t, t \in T\}.$$

The main goal of the paper is to investigate the geometrical properties of the solution set F , and special attention is paid to those systems that satisfy the so-called *locally Farkas-Minkowski* property (LFM, in short).

In linear semi-infinite programming, different characterizations of the dimension of the feasible and optimal sets were provided in [1]. For linear semi-infinite systems, the LFM property was introduced in [2], while [3] gave account of the most relevant properties of the systems enjoying this property. Chapter 5 in [3] is devoted to the geometry of their solution sets and, in particular, Theorem 5.9 there provides

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formulas for the dimension of F , for its affinity, and for its topological relative interior. In this theorem the interior set of F was characterized as the set of the Slater points of the system obtained from σ by elimination of all the trivial inequalities.

For convex semi-infinite systems, the *LFM property* is introduced in [3, Section 7.5], and its role as a *constraint qualification* for convex semi-infinite programming is emphasized there. In [4] the relationship between this constraint qualification and the upper semicontinuity (in the sense of Berge) of the so-called *active* and *sup-active mappings* is analyzed, as well as the fulfilment of the Valadier formula for the supremum function under some conditions involving the LFM property.

Section 2 of this paper studies the relative interior and the relative boundary of F for a general consistent convex system. We establish similar inclusions to those which are valid in the linear case ([3, Chapter 5]). Section 3 analyzes the relationship between the concepts of face and set of carrier indices for a linear representation of σ , σ_L , and their convex counterparts. In Section 4, some appealing geometrical properties of the solution set F of a convex LFM system are derived, and the main differences with the linear case are pointed out. Finally, in Section 5, different conditions are given guaranteeing a complete characterization of the interior and the relative interior of F .

Let us introduce the necessary notation. Given a non-empty set X of the Euclidean space \mathbb{R}^n , the *convex* (*conical*, *affine*, *linear*) *hull* of X is denoted by $\text{conv } X$ (*cone* X , *aff* X , *span* X , respectively), and X° represents the *polar cone* of X ,

$$X^\circ = \{y \in \mathbb{R}^n \mid \langle y, x \rangle \leq 0 \text{ for all } x \in X\}.$$

It is assumed that $\text{cone}(\emptyset) = \{0_n\}$, where 0_n is the null-vector in \mathbb{R}^n . We represent by $\dim X$ the dimension of $\text{aff } X$.

The largest subspace contained in the recession cone of X is called the *lineality space* of X , and is denoted by $\text{lin } X$. A convex cone is *pointed* if its lineality space is reduced to the null-vector.

From the topological side, $\text{int } X$, $\text{cl } X$ and $\text{bd } X$ represent the *interior*, the *closure* and the *boundary* of X , respectively, whereas $\text{rint } X$ and $\text{rbd } X$ represent the *relative interior* and the *relative boundary* of X (relatively to $\text{aff } X$), respectively.

If $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ is a l.s.c. proper convex function, the *effective domain*, and the *graph* of f are, respectively, the non-empty sets

$$\text{dom } f := \{x \in \mathbb{R}^n \mid f(x) < +\infty\},$$

and

$$\text{gph } f := \left\{ \begin{pmatrix} x \\ f(x) \end{pmatrix} \in \mathbb{R}^{n+1} \mid x \in \text{dom } f \right\}.$$

If $x \in \text{dom } f$, the *one-sided directional derivative* of f at x with respect to $v \in \mathbb{R}^n$

$$f'(x; v) = \lim_{\lambda \downarrow 0} \frac{f(x + \lambda v) - f(x)}{\lambda}$$

always exists ($+\infty$ and $-\infty$ being allowed as limits).

Finally, the *conjugate function* of f is the l.s.c. proper convex function defined by

$$f^*(u) = \sup \{ \langle u, x \rangle - f(x) \mid x \in \text{dom } f \}.$$

2. Preliminary results

Let $\sigma = \{f_t(x) \leq 0, t \in T\}$ be a consistent convex system. We associate with each index $t \in T$ the set

$$F_t := \{x \in F \mid f_t(x) = 0\}.$$

If σ is linear, then $F_t = \{x \in F \mid \langle a_t, x \rangle = b_t\}$ is an exposed face of F , but in the convex setting, F_t is not convex in general:

EXAMPLE 1. Let $\sigma = \{f_t(x) \leq 0, t \in T\}$ be the convex system, in \mathbb{R} , where $T = [1, 2]$ and

$$f_t(x) := x^2 - t.$$

The solution set is $F = [-1, 1]$, and

$$F_t = \begin{cases} \{-1, 1\}, & \text{if } t = 1, \\ \emptyset, & \text{if } t \in]1, 2]. \end{cases}$$

Obviously F_1 is not a convex set.

Nevertheless, the sets F_t enjoy the same property that every convex subset of a convex set C must verify in order to be a face of C [5, p.162].

PROPOSITION 1. *Let $\sigma = \{f_t(x) \leq 0, t \in T\}$ be a consistent convex system. For any $t \in T$, every closed line segment in F with a relative interior point in F_t is entirely contained in F_t .*

PROOF. Suppose that $[x, y] \subset F$ and $z \in]x, y[\cap F_t$. Consider $\alpha \in]0, 1[$ such that $z = \alpha x + (1 - \alpha)y$. Then, since x and y are in F ,

$$0 = f_t(z) \leq \alpha f_t(x) + (1 - \alpha)f_t(y) \leq 0,$$

which entails $f_t(x) = f_t(y) = 0$ and, so, x and y are also in F_t .

This proves that every closed line segment contained in F such that its relative interior intersects F_t , also verifies that its endpoints are in F_t . Now we apply this result to the following line segments:

Consider $\lambda \in]0, 1[$, $\lambda \neq \alpha$ and define $z_\lambda := \lambda x + (1 - \lambda)y$. If $\lambda > \alpha$, take the segment $[x, z_\lambda]$, and if $\lambda < \alpha$, take $[z_\lambda, y]$. In any case, $z_\lambda \in F_t$. \square

COROLLARY 1. *Let $\sigma = \{f_t(x) \leq 0, t \in T\}$ be a consistent convex system. For any $t \in T$, the following implication holds:*

$$F_t \cap \text{rint } F \neq \emptyset \Rightarrow F_t = F.$$

PROOF. Assume that $z \in F_t \cap \text{rint } F$, and take an arbitrary $x \in F$, $x \neq z$. Because $z \in \text{rint } F$, there will exist $\mu > 1$ such that $x_\mu := x + \mu(z - x) \in F$. Then $[x, x_\mu]$ is a closed line segment in F with a relative interior point, z , in F_t . Hence, Proposition 1 yields $x \in [x, x_\mu] \subset F_t$ and $F \subset F_t$. Because the other inclusion is obvious, $F = F_t$. \square

DEFINITION 1. If $f_t \equiv 0$, the corresponding inequality in σ is said to be trivial. We say that $t \in T$ is a proper index if f_t is not a constant function.

The index $t \in T$ is a carrier index in σ if $F = F_t$. The set of carrier indices is denoted by T_C .

Next we approach the relationship between $\text{rint } F$, $\text{rbd } F$, T_C and the sets F_t , $t \in T$.

PROPOSITION 2. Let $\sigma = \{f_t(x) \leq 0, t \in T\}$ be a consistent convex system, and assume that the following condition holds:

$$F \cap I \neq \emptyset, \quad (2.1)$$

where

$$I := \bigcap_{t \in T \setminus T_C} \text{rint}(\text{dom } f_t). \quad (2.2)$$

Then

$$\text{rint } F \subset \{x \in I \mid f_t(x) < 0, t \in T \setminus T_C\}. \quad (2.3)$$

PROOF. For $t \in T$, we consider the closed convex set

$$G_t = \{x \in \mathbb{R}^n \mid f_t(x) \leq 0\}.$$

G_t is non-empty because σ is consistent and, obviously, $F = \bigcap_{t \in T} G_t$.

Let $t \in T \setminus T_C$. Then, there exists $x^0 \in F$ such that $f_t(x^0) < 0$. Applying Theorem 7.6 in [5], we have

$$\text{rint } G_t = \{x \in \text{rint}(\text{dom } f_t) \mid f_t(x) < 0\},$$

and

$$\text{rbd } G_t = \{G_t \setminus \text{rint}(\text{dom } f_t)\} \cup \{x \in \mathbb{R}^n \mid f_t(x) = 0\}. \quad (2.4)$$

If $(\text{rint } F) \cap (\text{rint } G_t) = \emptyset$, and since $F \subset G_t$, it will be

$$\text{rint } F \subset \text{rbd } G_t. \quad (2.5)$$

Assume that there exists $z \in \text{rint } F$ such that $f_t(z) = 0$; in other words, that $F_t \cap \text{rint } F \neq \emptyset$. Then Corollary 1 yields $F_t = F$, and this contradicts $t \notin T_C$. From this consideration, together with (2.5) and (2.4), we get

$$\text{rint } F \subset G_t \setminus \text{rint}(\text{dom } f_t).$$

Since $G_t \setminus \text{rint}(\text{dom } f_t)$ is closed,

$$F = \text{cl } F = \text{cl}(\text{rint } F) \subset G_t \setminus \text{rint}(\text{dom } f_t),$$

and this contradicts the assumption (2.1).

Then we have concluded that $(\text{rint } F) \cap (\text{rint } G_t) \neq \emptyset$ and, by Theorem 6.5 in [5],

$$\text{rint } F = \text{rint}(F \cap G_t) = (\text{rint } F) \cap (\text{rint } G_t),$$

which entails

$$\text{rint } F \subset \text{rint } G_t. \quad (2.6)$$

□

When the functions f_t , $t \in T$, are all finite-valued, (2.1) is satisfied trivially with $I = \mathbb{R}^n$. On other hand, is clear that if the set I is empty, the inclusion (2.3) cannot hold. Next we provide a relaxation of (2.3), extending Theorem 5.1 in [3], and that does not require any additional condition as (2.1).

PROPOSITION 3. *Let $\sigma = \{f_t(x) \leq 0, t \in T\}$ be a consistent convex system. Then*

$$\text{rint } F \subset \{x \in \mathbb{R}^n \mid f_t(x) < 0, t \in T \setminus T_C; f_t(x) = 0, t \in T_C\}, \quad (2.7)$$

and

$$\text{rbd } F \supset \bigcup_{t \in T \setminus T_C} F_t. \quad (2.8)$$

PROOF. Let $t \in T \setminus T_C$ and $G_t = \{x \in \mathbb{R}^n \mid f_t(x) \leq 0\}$.

If $z \in \text{rint } F \subset G_t$ is such that $f_t(z) = 0$, Corollary 1 leads us to $F_t = F$, in contradiction with $t \notin T_C$. Thus

$$\text{rint } F \subset \{x \in \mathbb{R}^n \mid f_t(x) < 0\}. \quad (2.9)$$

Moreover, if $t \in T_C$,

$$\text{rint } F \subset F \subset \{x \in \mathbb{R}^n \mid f_t(x) = 0\}. \quad (2.10)$$

Then, (2.7) follows from (2.9) and (2.10), and (2.8) is a trivial consequence of (2.7), by complementarity. \square

REMARK 1. In the linear case, and according to Theorem 5.1 in [3], $\text{bd } F$ includes the union of all the faces F_t of proper indices; i.e., those indices such that $a_t \neq 0_n$.

In the convex case, the following example shows that we can find a proper index t such that F_t is not contained in $\text{bd } F$.

EXAMPLE 2. Let $\sigma = \{f_t(x) \leq 0, t \in T\}$ be the system, in \mathbb{R} , for which

$$f_t(x) := \max\{0, tx^2 - t\}, \quad T = [1, 2].$$

Observe that $F = [-1, 1] = F_t$, for all $t \in T$, whereas $\text{bd } F = \{-1, 1\}$.

Corollary 4, in the following section, establishes a sufficient condition that, in the convex case, guarantees the inclusion $F_t \subset \text{bd } F$, for any index t .

DEFINITION 2. *A solution x^0 of $\sigma = \{f_t(x) \leq 0, t \in T\}$ is said to be a Slater point of σ if it satisfies all the inequalities strictly; i.e., if $f_t(x^0) < 0$, for all $t \in T$.*

COROLLARY 2. *Let $\sigma = \{f_t(x) \leq 0, t \in T\}$ be a consistent convex system. Then σ has a Slater point if and only if $T_C = \emptyset$.*

PROOF. Let us start by assuming the existence of a Slater point; i.e., a point $x^0 \in F$ such that $f_t(x^0) < 0$, for all $t \in T$. Then $F \neq F_t$, for every $t \in T$, and $T_C = \emptyset$.

For the converse implication, if $T_C = \emptyset$, Proposition 3 allows us to write

$$\text{rint } F \subset \{x \in \mathbb{R}^n \mid f_t(x) < 0, t \in T\}.$$

Since F is a non-empty convex set, $\text{rint } F \neq \emptyset$ and there exists a Slater point for σ . \square

DEFINITION 3. An index $t \in T$ (or the constraint $f_t(x) \leq 0$) is active at $\bar{x} \in F$ if $f_t(\bar{x}) = 0$. The set of active indices at $\bar{x} \in F$ is

$$T(\bar{x}) = \{t \in T \mid f_t(\bar{x}) = 0\},$$

and the active cone at $\bar{x} \in F$ is

$$A(\bar{x}) := \text{cone} \left(\bigcup \{\partial f_t(\bar{x}), t \in T(\bar{x})\} \right),$$

where $\partial f_t(\bar{x})$ represents the subdifferential set of f_t at \bar{x} .

Remember that $\text{dom } f_t = \mathbb{R}^n$ implies that $\partial f_t(\bar{x})$ is a non-empty compact set [5, Theorem 23.4]. In the linear case, $A(\bar{x}) = \text{cone} \{a_t, t \in T(\bar{x})\}$.

DEFINITION 4. A consistent convex system $\sigma = \{f_t(x) \leq 0, t \in T\}$ is tight when $\dim A(\bar{x}) > 0$, for every $\bar{x} \in \text{bd } F$.

In the linear case, a tight system verifies that $\text{bd } F$ is the union of all the faces F_t of proper indices [3, p. 103]. Again, this property might fail for a convex system, as Example 2 shows. Instead of that, we have the following result:

PROPOSITION 4. Let $\sigma = \{f_t(x) \leq 0, t \in T\}$ be a tight convex system. Then $\text{bd } F$ is included in the union of all the sets F_t associated with proper indices.

PROOF. If $\bar{x} \in \text{bd } F$, then $\dim A(\bar{x}) > 0$, and hence, there exist $t \in T(\bar{x})$ and $v \in \partial f_t(\bar{x})$ such that $v \neq 0_n$. This implies that $\bar{x} \in F_t$ and t is a proper index (if f_t were a constant function, then $\partial f_t(\bar{x}) = \{0_n\}$). \square

The inclusion established in the last proposition can be strict, as Example 2 shows.

COROLLARY 3. Let $\sigma = \{f_t(x) \leq 0, t \in T\}$ be a tight convex system. If σ has a Slater point, then $\text{bd } F$ is the union of all the sets F_t , $t \in T$.

PROOF. By Corollary 2, $T_C = \emptyset$, and, by Proposition 3, $\text{bd } F$ includes the union of all the sets F_t , $t \in T$ (since $\text{bd } F \supset \text{rbd } F$). On the other hand, if $t \in T$ is not a proper index, then f_t will be a negative constant function (if $f_t \equiv 0$, then $t \in T_C$, and it cannot be positive, because σ is consistent), hence $F_t = \emptyset$. Since σ is tight, we get the statement thanks to Proposition 4. \square

3. Linear representation of a convex system

Let us associate with a convex system $\sigma = \{f_t(x) \leq 0, t \in T\}$ the following system of linear inequalities:

$$\sigma_L = \{\langle u, x \rangle \leq \langle u, y \rangle - f_t(y), t \in T, y \in \text{dom } \partial f_t, u \in \partial f_t(y)\}.$$

It is easy to verify that σ_L and σ have the same solution set, F , and, for this reason, it is said that σ_L and σ are *equivalent*. This provides a linear representation of F , whose indices set is

$$T_L := \{(t, y, u), t \in T, y \in \text{dom } \partial f_t, u \in \partial f_t(y)\}.$$

For each index $(t, y, u) \in T_L$, the associated exposed face will be denoted by $F_{(t, y, u)}$; i.e.,

$$F_{(t, y, u)} := \{x \in F \mid \langle u, x \rangle = \langle u, y \rangle - f_t(y)\}.$$

The set of carrier indices for σ_L is denoted by $T_{L,C}$.

Next we establish the relationships between the sets F_t , $F_{(t,y,u)}$, T_C and $T_{L,C}$.

PROPOSITION 5. *Let $\sigma = \{f_t(x) \leq 0, t \in T\}$ be a consistent convex system. Then, the following statements hold:*

- (i) *If $t \in T$, then $F_{(t,y,u)} \subset F_t$, for all $y \in \text{dom } \partial f_t$ and every $u \in \partial f_t(y)$.*
- (ii) *If $t \in T$, and $F \subset \text{dom } \partial f_t$, then*

$$F_t = \bigcup \{F_{(t,x,u)} \mid x \in F, u \in \partial f_t(x)\}. \quad (3.1)$$

- (iii) *If $F_t \neq \emptyset$ is an exposed face of F , and $F \subset \text{dom } \partial f_t$, there will exist $\bar{x} \in F$ and $\bar{u} \in \partial f_t(\bar{x})$ such that $F_t = F_{(t,\bar{x},\bar{u})}$.*
- (iv) *If $(t,y,u) \in T_{L,C}$, then $t \in T_C$. In particular $T_C = \emptyset$ entails $T_{L,C} = \emptyset$.*
- (v) *If $F \cap \text{rint}(\text{dom } f_t) \neq \emptyset$, for all $t \in T$, then $T_{L,C} = \emptyset$ implies $T_C = \emptyset$.*
- (vi) *If $F \cap \text{rint}(\text{dom } f_t) \neq \emptyset$, for all $t \in T$, and $\bar{t} \in T_C$, then there exists $x \in F$ and $u \in \partial f_{\bar{t}}(x)$ such that $(\bar{t}, x, u) \in T_{L,C}$.*
- (vii) *If $t \in T \setminus T_C$ and $F \subset \text{dom } \partial f_t$, then*

$$F_t = \bigcup \{F_{(t,x,u)} \mid x \in \text{rbd } F, u \in \partial f_t(x)\}.$$

PROOF. (i) If $F_{(t,y,u)} = \emptyset$ the inclusion is trivial. Otherwise, take $x \in F_{(t,y,u)}$. Since $x \in F$, we have $f_t(x) \leq 0$. Moreover,

$$f_t(x) \geq f_t(y) + \langle u, x - y \rangle = 0.$$

Then $f_t(x) = 0$, which implies that $x \in F_t$.

(ii) Since $F \subset \text{dom } \partial f_t$, and according to (i), $F_{(t,x,u)} \subset F_t$, for every $x \in F$ and all $u \in \partial f_t(x)$.

To show the converse inclusion, take $x \in F_t$. Since $x \in F$ and $f_t(x) = 0$, we consider the index (t, x, u) in T_L where $u \in \partial f_t(x) \neq \emptyset$. Then $\langle u, x \rangle = \langle u, x \rangle - f_t(x)$, and we have $x \in F_{(t,x,u)}$.

(iii) We know from (i) that $F_{(t,x,u)} \subset F_t$, for all $x \in F$ and every $u \in \partial f_t(x)$. We want to show that there exist $\bar{x} \in F$ and $\bar{u} \in \partial f_t(\bar{x})$ such that $F_t \subset F_{(t,\bar{x},\bar{u})}$.

If we prove that $(\text{rint } F_t) \cap F_{(t,\bar{x},\bar{u})} \neq \emptyset$ for some $(t, \bar{x}, \bar{u}) \in T_L$, Theorem 18.1 in [5] yields the inclusion we are looking for.

Hence, suppose that $(\text{rint } F_t) \cap F_{(t,x,u)} = \emptyset$ for all (t, x, u) , with $x \in F$ and $u \in \partial f_t(x)$. Then

$$(\text{rint } F_t) \cap \left(\bigcup \{F_{(t,x,u)} \mid x \in F, u \in \partial f_t(x)\} \right) = \emptyset,$$

and by (ii), we get $(\text{rint } F_t) \cap F_t = \emptyset$, and this is a contradiction because $F_t \neq \emptyset$.

(iv) If $(t, y, u) \in T_{L,C}$ then $F_{(t,y,u)} = F$ and by (i), $F \subset F_t$, hence $F = F_t$.

(v) If $T_{L,C} = \emptyset$, according to Corollary 5.1.1 in [3], σ_L has a Slater point. We shall prove that σ also has a Slater point, and Corollary 2 applies to conclude $T_C = \emptyset$.

Let $x^0 \in F$ be a Slater point of σ_L . Assume first that $x^0 \in \text{rbd } F$ and take $z \in \text{rint } F$. Then, for all $\lambda, 0 < \lambda < 1$, we can easily prove that $(1-\lambda)z + \lambda x^0 \in \text{rint } F$ is also a Slater point of σ_L .

Hence, we consider only the possibility $x^0 \in \text{rint } F$.

Now, Corollary 6.5.2 in [5] leads us to the inclusion $\text{rint } F \subset \text{rint}(\text{dom } f_t)$, for all $t \in T$, and, therefore, $\partial f_t(x^0) \neq \emptyset$, for all $t \in T$. If $f_t(x^0) = 0$ for some $t \in T$, we can take the index (t, x^0, u) in T_L (with $u \in \partial f_t(x^0)$). The associated constraint will be active at x^0 , and we shall get a contradiction.

(vi) Since $\bar{t} \in T_C$, $F_{\bar{t}} = F$. First, we shall show that $F \subset \text{dom } \partial f_{\bar{t}}$.

In the case $\dim F = 0$, there is nothing to prove, because $F \cap \text{rint}(\text{dom } f_{\bar{t}}) \neq \emptyset$ implies $F \subset \text{rint}(\text{dom } f_{\bar{t}}) \subset \text{dom } \partial f_{\bar{t}}$. Hence, let us suppose that $\dim F > 0$.

We know already that $\text{rint } F \subset \text{rint}(\text{dom } f_{\bar{t}}) \subset \text{dom } \partial f_{\bar{t}}$. If F were not included in $\text{dom } \partial f_{\bar{t}}$, there would exist $\bar{x} \in \text{rbd } F$ such that $\partial f_{\bar{t}}(\bar{x}) = \emptyset$. According to Theorem 23.3 in [5], for all $z \in \text{rint}(\text{dom } f_{\bar{t}})$,

$$f'_{\bar{t}}(\bar{x}; z - \bar{x}) = -\infty.$$

In particular, for all $z \in \text{rint } F$,

$$\lim_{\lambda \downarrow 0} \frac{f_{\bar{t}}(\bar{x} + \lambda(z - \bar{x}))}{\lambda} = -\infty, \quad (3.2)$$

because $f_{\bar{t}}(\bar{x}) = 0$. But, for $0 < \lambda \leq 1$,

$$\bar{x} + \lambda(z - \bar{x}) = (1 - \lambda)\bar{x} + \lambda z \in \text{rint } F,$$

and

$$\frac{f_{\bar{t}}(\bar{x} + \lambda(z - \bar{x}))}{\lambda} = 0,$$

contradicting (3.2). Hence, $F \subset \text{dom } \partial f_{\bar{t}}$.

By (iii), we get $F_{\bar{t}} = F = F_{(\bar{t}, x, u)}$, for some $x \in F$ and some $u \in \partial f_{\bar{t}}(x)$.

(vii) For every $t \in T$, (ii) entails

$$\bigcup \{F_{(t, x, u)} \mid x \in \text{rbd } F, u \in \partial f_t(x)\} \subset F_t.$$

For $t \in T \setminus T_C$, (2.8) yields $F_t \subset \text{rbd } F$. Hence, if $\bar{x} \in F_t$, we have $\bar{x} \in F_{(t, \bar{x}, \bar{u})}$, for every $\bar{u} \in \partial f_t(\bar{x})$. \square

COROLLARY 4. *Let $\sigma = \{f_t(x) \leq 0, t \in T\}$ be a consistent convex system. Then*

$$\text{bd } F \supset \bigcup_{t \in T} F_t,$$

provided that one of the following conditions holds:

- (i) $T_C = \emptyset$;
- (ii) $T_C \neq \emptyset$, $F \cap \text{rint}(\text{dom } f_t) \neq \emptyset$, for all $t \in T$, and $\inf f_t < 0$ for each $t \in T_C$.

PROOF. (i) If $T_C = \emptyset$, then the statement is true according to (2.8).

(ii) Take $t \in T \setminus T_C$. Then, again by (2.8), $F_t \subset \text{rbd } F \subset \text{bd } F$.

Take now, $t \in T_C$. We shall show that $\text{bd } F = F = F_t$.

Since $t \in T_C$, by Proposition 5 (vi), there exist $x \in F$ and $u \in \partial f_t(x)$ such that $(t, x, u) \in T_{L,C}$. Then

$$F \subset \{y \in \mathbb{R}^n \mid \langle u, y \rangle = \langle u, x \rangle - f_t(x)\}.$$

But $f_t(x) = 0$, and $0_n \notin \partial f_t(x)$ (because $\inf f_t < 0$). Then we have

$$F \subset \{y \in \mathbb{R}^n \mid \langle u, y \rangle = \langle u, x \rangle\},$$

with $u \neq 0_n$. This implies $\text{int } F = \emptyset$ and, hence, $\text{bd } F = F$. \square

REMARK 2. The condition $F \subset \text{dom } \partial f_t$ cannot be suppressed in Proposition 5 (ii) and (iii), as the following examples show.

EXAMPLE 3. Let us consider a convex system, in \mathbb{R} , with a unique inequality, $\sigma = \{f_0(x) \leq 0\}$, where

$$f_0(x) := \begin{cases} -(1-x^2)^{\frac{1}{2}}, & \text{if } |x| \leq 1, \\ +\infty & \text{if } |x| > 1. \end{cases}$$

Observe that $F = [-1, 1]$. The function f_0 is subdifferentiable (in fact, differentiable) at x , when $|x| < 1$, but $\partial f_0(x) = \emptyset$ if $|x| = 1$. We get $F_0 = \{-1, 1\}$ and, for all $x \in F \cap \text{dom } \partial f_0 =]-1, 1[$,

$$\partial f_0(x) = \{\nabla f_0(x)\} = \left\{x(1-x^2)^{-\frac{1}{2}}\right\}.$$

Then, for all $x \in]-1, 1[$, we have

$$F_{(0,x,\nabla f_0(x))} = \{y \in F \mid yx = 1\} = \emptyset$$

and, consequently,

$$\bigcup \{F_{(0,x,u)} \mid x \in F \cap \text{dom } \partial f_0, u \in \partial f_0(x)\} = \emptyset \neq F_1.$$

EXAMPLE 4. Let $\sigma = \{f_0(x) \leq 0\}$ be a convex system, in \mathbb{R} , with

$$f_0(x) := \begin{cases} -x^{\frac{1}{2}}, & \text{if } x \geq 0, \\ +\infty & \text{if } x < 0. \end{cases}$$

We have $F = [0, +\infty[$. The function f_0 is differentiable at $x > 0$, with $\nabla f_0(x) = -(1/2)x^{-(1/2)}$, but $\partial f_0(0) = \emptyset$. Moreover, $F_0 = \{0\}$, is an exposed face of F , and for all $x > 0$,

$$F_{(0,x,\nabla f_0(x))} = \{y \in F \mid y = -x\} = \emptyset.$$

REMARK 3. Finally, in Proposition 5, if $F \subset \text{rbd}(\text{dom } f_t)$, for some $t \in T$, (v) can fail, as we can see in the following example:

EXAMPLE 5. Let $\sigma = \{f_t(x) \leq 0, t \in T\}$ be the convex system, in \mathbb{R} , where $T = \{0\} \cup \mathbb{N}$, and

$$f_0(x) := \begin{cases} -(1-x^2)^{\frac{1}{2}}, & \text{if } |x| \leq 1, \\ +\infty & \text{if } |x| > 1. \end{cases}$$

$$f_r(x) := |x-1| - \frac{1}{r}, \quad r \in \mathbb{N}.$$

We have $F = \{1\} \subset \text{rbd}(\text{dom } f_0)$, and $T_C = \{0\}$.

In this case σ_L is the system composed by the following inequalities (without repetition):

$$\left\{ tx \leq 1, |t| < 1, x \leq 1 + \frac{1}{s}, s > 0, -x \leq -1 + \frac{1}{u}, u > 0 \right\},$$

and we conclude $T_{L,C} = \emptyset$.

This example also shows that (vi) could fail if $F \cap \text{rint}(\text{dom } f_{\bar{t}}) = \emptyset$, for some $\bar{t} \in T$ (here $\bar{t} = 0$).

4. Locally Farkas-Minkowski systems

DEFINITION 5. A consistent convex system σ is locally Farkas-Minkowski (LFM) if

$$D(F, \bar{x})^\circ = A(\bar{x}), \text{ for all } \bar{x} \in F,$$

where $D(F, \bar{x})$ is the cone of feasible directions to F at \bar{x} ; i.e.

$$D(F, \bar{x}) := \{y \in \mathbb{R}^n \mid \bar{x} + \mu y \in F, \text{ for some } \mu > 0\},$$

and $A(\bar{x})$ is the active cone at \bar{x} introduced in Definition 3.

Observe that $D(F, \bar{x})^\circ$ is nothing else than the *normal cone to F at \bar{x}* , also represented by $N(F, \bar{x})$. As a consequence of Lemma 2.1 in [4], the LFM property should be investigated only at the boundary feasible points (if σ is LFM, one has $A(\bar{x}) \neq \{0_n\}$ for every $\bar{x} \in \text{bd}(F)$; i.e., σ is tight). Also, it is shown in Theorem 7.10 in [3] that σ is LFM if and only if σ_L is LFM.

This property plays an important role as a constraint qualification in ordinary non-linear programming, where it is called *Basic Constraint Qualification* (BCQ, in brief). See, for instance [6, pp. 307-309]. In convex semi-infinite programming, its role as constraint qualification has been proved in [3, Theorem 7.8], and its relationship with the Slater condition appears in [3, Theorem 7.9]. Li, Nahak and Singer, in [7], give characterizations of the LFM property (using the term BCQ) through the closedness of certain associated convex cone-valued mappings. Also, assuming the BCQ, formulas for the distance of a point to the solution set of a semi-infinite system of convex inequalities are given.

Our principal objective is to provide a deeper knowledge of the geometrical behaviour of the LFM convex systems. For the sake of simplicity, in this section we shall consider only finite-valued convex functions; i.e., $\text{dom } f_t = \mathbb{R}^n$, for all $t \in T$ (see, for instance, [7]).

The next Theorem provides convex counterparts of the results established in [3, Theorem 5.9].

THEOREM 1. The following statements are valid for any LFM convex system $\sigma = \{f_t(x) \leq 0, t \in T\}$:

(i) For every $\bar{x} \in F$

$$\text{lin } A(\bar{x}) = (F - \bar{x})^\perp = \text{span} \{\cup_{t \in T_C} \cap_{y \in F} \partial f_t(y)\}. \quad (4.1)$$

(ii) If $T_C \neq \emptyset$, then

$$\text{aff } F = \{x \in \mathbb{R}^n \mid \langle u, x \rangle = \langle u, y \rangle, u \in \cup_{t \in T_C} \cap_{y \in F} \partial f_t(y)\}. \quad (4.2)$$

(iii) If $T_C = \emptyset$, then $\text{int } F$ is the set of Slater points of σ .

- (iv) If $\langle a, x \rangle \leq b$ is a supporting half-space to F defining an exposed face E , then $a \in A(y)$ for all $y \in E$, and E is contained in the intersection of a finite number of sets F_t associated with proper indices.

PROOF. (i) Given $\bar{x} \in F$, and denoting by $A_L(\bar{x})$ the active cone at \bar{x} corresponding to σ_L , we have $A(\bar{x}) = A_L(\bar{x})$, according to the proof of Theorem 7.10 in [3]. Then, Theorem 5.9 (i) in [3], yields

$$\text{lin } A(\bar{x}) = (F - \bar{x})^\perp = \text{span} \{u \in \mathbb{R}^n \mid (t, y, u) \in T_{L,C}\}. \quad (4.3)$$

We use the notation

$$X := \{u \in \mathbb{R}^n \mid (t, y, u) \in T_{L,C}\},$$

and

$$Y := \cup_{t \in T_C} \cap_{y \in F} \partial f_t(y).$$

First, we shall see that $X \subset Y$.

If $u \in X$, there exist $y \in \mathbb{R}^n$ and $t \in T$ such that $u \in \partial f_t(y)$ and $(t, y, u) \in T_{L,C}$. By Proposition 5 (iv), $t \in T_C$. Now we have to prove that $u \in \partial f_t(x)$ for all $x \in F$. Since $F_{(t,y,u)} = F$, $f_t(y) = \langle u, y - x \rangle$, for all $x \in F$.

On the other hand, for all $z \in \mathbb{R}^n$, $f_t(z) \geq f_t(y) + \langle u, z - y \rangle$. Replacing $f_t(y)$ by $\langle u, y - x \rangle$, we obtain

$$f_t(z) \geq \langle u, y - x \rangle + \langle u, z - y \rangle = \langle u, z - x \rangle,$$

and, since $f_t(x) = 0$, we can write, for all $z \in \mathbb{R}^n$,

$$f_t(z) \geq f_t(x) + \langle u, z - x \rangle,$$

for all $x \in F$, and we get $u \in \partial f_t(x)$ for all $x \in F$. Thus $u \in Y$ and we have $\text{span } X \subset \text{span } Y$.

To see the opposite inclusion, we prove that $\text{span } Y \subset (F - \bar{x})^\perp = \text{span } X$, according to (4.3).

Take $v \in Y$. Then there exists $t \in T_C$ such that $v \in \partial f_t(y)$, for all $y \in F$. Since $v \in \partial f_t(\bar{x})$, $f_t(y) \geq f_t(\bar{x}) + \langle v, y - \bar{x} \rangle$ for all $y \in F$, and similarly, $f_t(\bar{x}) \geq f_t(y) + \langle v, \bar{x} - y \rangle$ for all $y \in F$. Taking into account that $f_t(x) = 0$, for all $x \in F$, we get $v \in (F - \bar{x})^\perp$.

(ii) First, observe that for $u \in Y$ and according to (i), $\langle u, y \rangle$ is constant, for all $y \in F$. Consequently, the equalities in (4.2) are well defined.

If $T_C \neq \emptyset$, then $T_{L,C} \neq \emptyset$, and by Theorem 5.9 (iii) in [3] we have

$$\text{aff } F = \{x \in \mathbb{R}^n \mid \langle u, x \rangle = \langle u, y \rangle - f_t(y), (t, y, u) \in T_{L,C}\}. \quad (4.4)$$

Denoting

$$Z := \{x \in \mathbb{R}^n \mid \langle u, x \rangle = \langle u, y \rangle, u \in \cup_{t \in T_C} \cap_{y \in F} \partial f_t(y)\},$$

we shall see first that $\text{aff } F \subseteq Z$.

Take $x \in \text{aff } F$. Then, by (4.4), $\langle u, x \rangle = \langle u, y \rangle - f_t(y)$ for all $(t, y, u) \in T_{L,C}$. If $t \in T_C$ and $u \in \cap_{y \in F} \partial f_t(y)$, then $(t, z, u) \in T_{L,C}$, for all $z \in F$ (otherwise, there would exist $z \in F$ such that $(t, z, u) \in T_L \setminus T_{L,C}$, and this implies the existence of $z^0 \in F$ verifying $\langle u, z^0 \rangle < \langle u, z \rangle - f_t(z)$. But $f_t(z) = 0$, hence $\langle u, z^0 \rangle < \langle u, z \rangle$,

contradicting the fact that $\langle u, w \rangle$ is constant for all $w \in F$. We conclude that $\langle u, x \rangle = \langle u, z \rangle$, for all $z \in F$ and $x \in Z$.

Now, take $x \in Z$, and $(t, y, u) \in T_{L,C}$. It has been shown in (i) that $t \in T_C$ and $u \in \partial f_t(z)$ for all $z \in F$; hence $\langle u, x \rangle = \langle u, z \rangle$, for all $z \in F$. On the other hand, $F_{(t,y,u)} = F$, and for all $z \in F$, we have $\langle u, z \rangle = \langle u, y \rangle - f_t(y)$. We conclude that $\langle u, x \rangle = \langle u, y \rangle - f_t(y)$, and therefore $x \in \text{aff } F$.

(iii) If $T_C = \emptyset$, then $T_{L,C} = \emptyset$, according to Proposition 5 (v). Thus there are no trivial inequalities in σ_L , and according to Theorem 5.9 (iv) in [3], $\text{int } F$ can be expressed in the form

$$\{x \in \mathbb{R}^n \mid \langle u, x \rangle < \langle u, y \rangle - f_t(y), (t, y, u) \in T_L\}.$$

We shall prove that this is the set of the Slater points of σ .

First, take $x \in F$ such that $f_t(x) < 0$, for every $t \in T$. Now, if $(t, y, u) \in T_L$, then $0 > f_t(x) \geq f_t(y) + \langle u, x - y \rangle$; hence $\langle u, x \rangle < \langle u, y \rangle - f_t(y)$. On the other hand, $\text{int } F$ is contained in the set of the Slater points of σ (which is not empty, according to Corollary 2), by (2.7).

(iv) Let $E = F \cap \{x \in \mathbb{R}^n \mid \langle a, x \rangle = b\}$ and suppose that $\langle a, x \rangle \leq b$ for all $x \in F$. By Theorem 5.9 (v) in [3], $a \in A(y)$, for all $y \in E$, and

$$E = \bigcap_{i \in I} F_{(t_i, y_i, u_i)},$$

with $u_i \neq 0_n$, for all $i \in I$, and I finite. Each $F_{(t_i, y_i, u_i)}$ is contained in F_{t_i} , where f_{t_i} is not a constant function (otherwise $\partial f_{t_i}(x) = 0_n$, for all x), and then t_i is a proper index, for every $i \in I$. \square

COROLLARY 5. Let $\sigma = \{f_t(x) \leq 0, t \in T\}$ be a LFM convex system. Then:

(i) $\dim F = n$ if and only if $A(x)$ is pointed at every $x \in F$, and this happens if

$$0_{n+1} \notin \text{conv} \left(\bigcup \{ \text{gph } f_t^*, t \in T \} \setminus \{0_{n+1}\} \right). \quad (4.5)$$

(ii) $\dim F = n - \dim \text{span} \{ \bigcup_{t \in T_C} \cap_{y \in F} \partial f_t(y) \}$.

PROOF. (i) The result is a straightforward consequence of Theorem 5.9 (ii) in [3], since $A_L(x) = A(x)$, for all $x \in F$.

Concerning the second statement, again from Theorem 5.9 (ii) in [3], $A(x)$ is pointed at every $x \in F$ if and only if

$$0_{n+1} \notin \text{conv} \left\{ \begin{pmatrix} u \\ \langle u, y \rangle - f_t(y) \end{pmatrix} \neq 0_{n+1}, (t, y, u) \in T_L \right\}. \quad (4.6)$$

By Theorem 23.5 in [5], $u \in \partial f_t(y)$ is equivalent to $f_t(y) + f_t^*(u) = \langle u, y \rangle$, hence (4.6) can be rewritten

$$0_{n+1} \notin \text{conv} \left\{ \begin{pmatrix} u \\ f_t^*(u) \end{pmatrix} \neq 0_{n+1}, u \in \text{range } \partial f_t, t \in T \right\},$$

where

$$\text{range } \partial f_t := \bigcup \{ \partial f_t(x) \mid x \in \mathbb{R}^n \},$$

and from the inclusion $\text{range } \partial f_t \subset \text{dom } f_t^*$ ([5, p.227]), we get the sufficient condition (4.5).

(ii) It is a direct consequence of (4.1). \square

REMARK 4. If σ is not LFM, the three subspaces which appear in (4.1) always verify the inclusions:

$$\begin{aligned} \text{span}\{\cup_{t \in T_C} \cap_{y \in F} \partial f_t(y)\} &\subset (F - \bar{x})^\perp, \\ \text{lin } A(\bar{x}) &\subset (F - \bar{x})^\perp. \end{aligned}$$

But these inclusions can be strict, and the subspaces $\text{span}\{\cup_{t \in T_C} \cap_{y \in F} \partial f_t(y)\}$ and $\text{lin } A(\bar{x})$ do not coincide in general, as the following example shows.

EXAMPLE 6. Let us consider the linear consistent system, defined in \mathbb{R}^2 ,

$$\sigma = \{x_1 \leq 0, -x_1 \leq t, x_2 \leq t, -x_2 \leq t, t \in]0, 1]\}.$$

Then $F = \{0_2\}$, but σ is not LFM in F , since $D(F, 0_2)^\circ = \mathbb{R}^2$ and $A(0_2) = \left\{ \begin{pmatrix} \lambda \\ 0 \end{pmatrix}, \lambda \geq 0 \right\}$. In this case, $(F - 0_2)^\perp = \mathbb{R}^2$ and $\text{lin } A(0_2) = \{0_2\}$, whereas $\text{span}\{a_t, t \in T_C\} = \left\{ \begin{pmatrix} \lambda \\ 0 \end{pmatrix}, \lambda \in \mathbb{R} \right\}$

REMARK 5. Applying Theorem 5.9 (iv) in [3] to σ_L , it can be stated that $\text{int } F$ is the set of the Slater points of the (equivalent) system obtained by the elimination of the trivial inequalities in σ_L . This is not true, in general, for the convex system σ .

EXAMPLE 7. Let us consider again the system in Example 2. We get $F = [-1, 1] = F_t$, for all $t \in T$; then $T_C = T$. The system σ is LFM and the set of Slater points is empty, but $\text{int } F \neq \emptyset$.

REMARK 6. Again by Theorem 5.9 (iv) in [3], when it is applied to σ_L , it holds that $\text{rint } F$ is the solution set of the system

$$\left\{ \begin{array}{l} \langle u, x \rangle < \langle u, y \rangle - f_t(y), \quad (t, y, u) \in T_L \setminus T_{L,C}, \\ \langle u, x \rangle = \langle u, y \rangle - f_t(y), \quad (t, y, u) \in T_{L,C} \end{array} \right\}, \quad (4.7)$$

but, in general,

$$\text{rint } F = \{x \in \mathbb{R}^n \mid f_t(x) < 0, t \in T \setminus T_C; f_t(x) = 0, t \in T_C\},$$

does not hold, as the following example shows.

EXAMPLE 8. Let $\sigma = \{f_t(x) \leq 0, t \in T\}$ be a convex system, in \mathbb{R} , where $T = \{0\} \cup [2, 3]$ and

$$\begin{aligned} f_0(x) &:= \begin{cases} 0, & \text{if } x \leq 0, \\ x, & \text{if } x > 0. \end{cases} \\ f_t(x) &:= e^x - t, \quad t \in [2, 3]. \end{aligned}$$

We observe that $F =]-\infty, 0]$, the system σ is LFM, and $T_C = \{0\}$. Nevertheless in this case, and since $F_t = \emptyset$, for all $t \in [2, 3]$,

$$\text{rint } F = \text{int } F \subsetneq F = \{x \in \mathbb{R} \mid f_t(x) < 0, t \in [2, 3]; f_0(x) = 0\}.$$

REMARK 7. Theorem 5.9 (v) in [3] shows that, in the linear case, if a supporting half-space to F defines an exposed face E , then E is the intersection of a finite number of faces F_t of proper indices. In the convex case, we can only guarantee that E is contained in the intersection of a finite number of sets F_t of proper indices (Theorem 1 (iv)), but we cannot ensure the equality.

EXAMPLE 9. Let us consider again the system in Example 8. Take $E = \{0\}$. Every index is proper, and $F_t = \emptyset$, for all $t \in [2, 3]$, $F_0 = F$, and $E \subsetneq F_0$.

5. Characterization of the interior and the relative interior of the solution set

As it was pointed out in Remark 5, if $\sigma = \{f_t(x) \leq 0, t \in T\}$ is a consistent LFM convex system, it is not guaranteed the equality

$$\text{int } F = \left\{x \in \mathbb{R}^n \mid f_t(x) < 0, t \in \tilde{T}\right\}, \quad (5.1)$$

where $\tilde{T} = \{t \in T \mid f_t \text{ is not identically zero}\}$; i.e, the indices set of the equivalent system $\tilde{\sigma}$ obtained by the elimination of all the trivial inequalities in σ .

Theorem 2 will provide sufficient and necessary conditions for the fulfilment of the equality (5.1). Its proof requires the following lemma.

LEMMA 1. *Let $\sigma = \{f_t(x) \leq 0, t \in T\}$ be a consistent convex system and assume that $T_C \neq \emptyset$. If there exists $t \in T_C$ such that $\inf f_t < 0$, then $\text{int } F = \emptyset$.*

PROOF. Since $t \in T_C$, and by Proposition 5 (vi), $F = F_{(t,y,u)}$, for some $y \in F$ and $u \in \partial f_t(y)$; hence $F \subset \{x \in \mathbb{R}^n \mid \langle u, x \rangle = \langle u, y \rangle\}$. If $u = 0_n$, then y is a global minimum of f_t , but $f_t(y) = 0$ and $\inf f_t < 0$. We conclude that $u \neq 0_n$, which implies that $\dim F < n$ and $\text{int } F = \emptyset$. \square

THEOREM 2. *Let $\sigma = \{f_t(x) \leq 0, t \in T\}$ be a LFM convex system and let $\tilde{\sigma}$ be the equivalent system obtained by the elimination of the trivial inequalities. Each of the following conditions is sufficient for σ to verify (5.1):*

- (i) $\tilde{T}_C = \emptyset$;
- (ii) $\tilde{T}_C \neq \emptyset$ and there exists $\tilde{t} \in \tilde{T}_C$ such that $\inf f_{\tilde{t}} < 0$;
- (iii) $\tilde{T}_C \neq \emptyset$ and there exists $\tilde{t} \in \tilde{T}_C$ such that $f_{\tilde{t}}$ is not differentiable at any point in F .

Moreover, if σ verifies the equality (5.1), then one of the conditions (i)-(iii) must hold.

PROOF. (i) If $\tilde{T}_C = \emptyset$, since F is the solution set of $\tilde{\sigma}$, the result follows from Theorem 1 (iii).

We proceed now in the following way: assuming that $\tilde{T}_C \neq \emptyset$, then $T_C \neq \emptyset$ and for all $t \in T_C$, f_t is not identically zero. Moreover the set of Slater points of $\tilde{\sigma}$ is empty. We shall analyze all the possibilities, and we shall see that only in the cases specified in (ii) and (iii), it happens that $\text{int } F = \emptyset$. Therefore we shall have shown the direct and the converse statements.

Case 1 (which corresponds with (ii)).

If there exists $\tilde{t} \in \tilde{T}_C$ such that $\inf f_{\tilde{t}} < 0$, then $\text{int } F = \emptyset$, by Lemma 1.

Case 2.

For every $t \in T_C$, $\inf f_t = 0$. This is equivalent to $0_n \in \partial f_t(z)$, for all $z \in F$, and every $t \in T_C$, then $0_n \in \cap_{z \in F} \partial f_t(z)$, for all $t \in T_C$.

By Corollary 5 (ii),

$$\dim F = n - \dim \text{span} \{ \cup_{t \in T_C} \cap_{z \in F} \partial f_t(z) \},$$

and two possibilities may occur:

Case 2.1.

For every $t \in T_C$, f_t is differentiable at some point $z_t \in F$, then $\{0_n\} = \partial f_t(z_t)$, which implies that $\cup_{t \in T_C} \cap_{z \in F} \partial f_t(z) = \{0_n\}$. Hence, $\dim F = n$ and $\text{int } F \neq \emptyset$. In this case, the set of Slater points of $\tilde{\sigma}$ does not agree with $\text{int } F$.

Case 2.2 (which corresponds with (iii)).

There exists $\tilde{t} \in T_C$ such that $f_{\tilde{t}}$ is not differentiable at any point in F . Then, by Theorem 25.5 in [5], the set of points $D \subset \mathbb{R}^n$ where $f_{\tilde{t}}$ is differentiable is a dense subset of \mathbb{R}^n , and we have to conclude that $\text{int } F = \emptyset$ (otherwise $D \cap \text{int } F \neq \emptyset$). \square

Next, we shall characterize the relative interior of the solution set F . The objective is, again, to provide sufficient and necessary conditions for the equality

$$\text{rint } F = \{x \in \mathbb{R}^n \mid f_t(x) < 0, t \in T \setminus T_C; f_t(x) = 0, t \in T_C\}, \quad (5.2)$$

which is valid in the linear case, but not, in general, in the convex one (remember Remark 6).

We shall assume that F is not a singleton. Otherwise, $F = \{\bar{x}\}$, and then $\text{rint } F = F$, and $f_t(\bar{x}) = 0$ if and only if $t \in T_C$, while $f_t(\bar{x}) < 0$ if and only if $t \in T \setminus T_C$, in which case, $F_t = \emptyset$. Hence the equality (5.2) trivially holds.

We start with a couple of necessary conditions (not involving the LFM property for σ), being the second one also sufficient when $\dim F = n$.

PROPOSITION 6. *Let $\sigma = \{f_t(x) \leq 0, t \in T\}$ be a consistent convex system. If the equality (5.2) holds, then there exists $t \in T \setminus T_C$ such that $F_t \neq \emptyset$.*

PROOF. Suppose that, for all $t \in T \setminus T_C$, $F_t = \emptyset$. Then for every $x \in \text{rbd } F$, $f_t(x) < 0$, if $t \in T \setminus T_C$, and $f_t(x) = 0$, if $t \in T_C$. Taking (2.7) into account, we conclude that

$$\text{rint } F \subsetneq \{x \in \mathbb{R}^n \mid f_t(x) < 0, t \in T \setminus T_C; f_t(x) = 0, t \in T_C\},$$

contradicting that (5.2) holds. \square

The following example shows that the necessary condition in Proposition 6 is not sufficient, even in the case that $F_t \neq \emptyset$, for all $t \in T \setminus T_C$. Moreover, in this example, σ is LFM.

EXAMPLE 10. Let us consider the system, in \mathbb{R} , $\sigma = \{f_t(x) \leq 0, t \in T\}$, where $T = [1, +\infty[$, and

$$f_1(x) := \begin{cases} -3x - 6, & \text{if } x \leq -2, \\ 0, & \text{if } -2 < x \leq 0, \\ x, & \text{if } x > 0. \end{cases}$$

$$f_t(x) := x^2 + (2-t)x - 2t, \quad t > 1.$$

Observe that $F = [-2, 0]$, $T_C = \{1\}$, $F_t = \{-2\}$, for all $t > 1$, and

$$\text{int } F \subsetneq]-2, 0] = \{x \in \mathbb{R} \mid f_t(x) < 0, t > 1; f_1(x) = 0\}.$$

PROPOSITION 7. *Let $\sigma = \{f_t(x) \leq 0, t \in T\}$ be a consistent convex system. If the equality (5.2) holds, then for all $\bar{x} \in \text{rbd } F$ there exists $t \in T(\bar{x})$ such that $\inf f_t < 0$. Moreover, if $\dim F = n$, the converse statement is also true.*

PROOF. It is evident from (2.7) that the equality (5.2) will hold if and only if for all $x \in \text{rbd } F$,

$$T(x) \cap (T \setminus T_C) \neq \emptyset. \quad (5.3)$$

If there exists $\bar{x} \in \text{rbd } F$ such that $\inf f_t = 0$, for all $t \in T(\bar{x})$, then

$$0 = f_t(\bar{x}) \leq f_t(x) \leq 0,$$

for every $x \in F$, and we get $T(\bar{x}) \subset T_C$, contradicting (5.3).

Now, for the converse statement, suppose that $\dim F = n$. We shall prove that (5.3) holds, for all $x \in \text{bd } F$.

Take $\bar{x} \in \text{bd } F$, and $t \in T(\bar{x})$ such that $\inf f_t < 0$. The set

$$G_t = \{x \in \mathbb{R}^n \mid f_t(x) \leq 0\},$$

is a full-dimensional closed convex set in \mathbb{R}^n , containing F , which verifies $\text{int } G_t = \{x \in \mathbb{R}^n \mid f_t(x) < 0\}$ and $\text{bd } G_t = \{x \in \mathbb{R}^n \mid f_t(x) = 0\}$ ([5, Theorem 7.6]). Since $\dim F = n$, $F \cap \text{int } G_t \neq \emptyset$ and there exists $x^0 \in F$ such that $f_t(x^0) < 0$. Thus $t \notin T_C$. \square

According to Proposition 7, the equality (5.2) does not hold in Example 10 because, for $\bar{x} = 0 \in \text{bd } F$, $\inf f_t = 0$, if $t \in T(\bar{x}) = \{1\}$.

Next example will show that, in the case $\dim F < n$, the necessary condition stated in Proposition 7 is not sufficient, even if the system σ is LFM.

EXAMPLE 11. Let $\sigma = \{f_t(x_1, x_2) \leq 0, t \in T\}$ be a convex system, in \mathbb{R}^2 , where $T = [0, 1] \cup \{2\}$ and

$$\begin{aligned} f_0(x_1, x_2) &:= |x_1| + [x_2]_+ \equiv |x_1| + \max\{0, x_2\}, \\ f_t(x_1, x_2) &:= x_1^2 + tx_2 - t, \quad t \in]0, 1], \\ f_2(x_1, x_2) &:= x_1. \end{aligned}$$

The solution set is $F = \{0\} \times]-\infty, 0]$ and $T_C = \{0, 2\} = T(0_2)$. We also have $\inf f_2 < 0$. We can see that $\{0_2\} = \text{rbd } F$ and $T(0_2) \cap (T \setminus T_C) = \emptyset$. Hence, the equality (5.2) does not hold.

Next we prove that σ is LFM.

Denoting by $g_1(x_1, x_2) = |x_1|$ and $g_2(x_1, x_2) = [x_2]_+$, we know that for $\bar{x} = 0_2$, $T(\bar{x}) = \{0, 2\}$ and

$$\begin{aligned} \partial f_0(\bar{x}) &= \partial g_1(\bar{x}) + \partial g_2(\bar{x}), \\ \partial g_1(\bar{x}) &= \{u \in \mathbb{R}^2 \mid -1 \leq u_1 \leq 1, u_2 = 0\}, \\ \partial g_2(\bar{x}) &= \{u \in \mathbb{R}^2 \mid u_1 = 0, 0 \leq u_2 \leq 1\}. \end{aligned}$$

Thus $\partial f_0(\bar{x}) = \{u \in \mathbb{R}^2 \mid -1 \leq u_1 \leq 1, 0 \leq u_2 \leq 1\}$.

On the other hand, $\partial f_2(\bar{x}) = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$. Then

$$A(\bar{x}) = \text{cone} \{ \partial f_0(\bar{x}) \cup \partial f_2(\bar{x}) \} = \mathbb{R} \times [0, +\infty[= D(F, \bar{x})^\circ.$$

Also, for $\bar{x} = \begin{pmatrix} 0 \\ x_2 \end{pmatrix}$, $x_2 < 0$, we get $T(\bar{x}) = \{0, 2\}$, and

$$\partial g_1(\bar{x}) = \{u \in \mathbb{R}^2 \mid -1 \leq u_1 \leq 1, u_2 = 0\}, \quad \partial g_2(\bar{x}) = \{0_2\}.$$

Then $\partial f_0(\bar{x}) = \{u \in \mathbb{R}^2 \mid -1 \leq u_1 \leq 1, u_2 = 0\}$ and $\partial f_2(\bar{x}) = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$. We obtain

$$A(\bar{x}) = \text{cone} \{ \partial f_0(\bar{x}) \cup \partial f_2(\bar{x}) \} = \mathbb{R} \times \{0\} = D(F, \bar{x})^\circ.$$

Except in the case $\dim F = n$, there is no relationship between both necessary conditions stated in Propositions 6 and 7:

(C.1) There exists an index $t \in T \setminus T_C$ such that $F_t \neq \emptyset$;

(C.2) for all $\bar{x} \in \text{rbd } F$, there exists $t \in T(\bar{x})$ such that $\inf f_t < 0$.

In fact, if $\dim F = n$, (C.2) is equivalent to the fulfilment of the equality (5.2), and it implies (C.1). But (C.1) does not imply (C.2), as Example 10 shows.

On the other hand, if $\dim F < n$, the system in Example 11 verifies (C.2), but for all $t \in T \setminus T_C =]0, 1]$, $F_t = \emptyset$, and therefore (C.1) fails. Finally, the following example shows that (C.1) does not imply (C.2).

EXAMPLE 12. Let $\sigma = \{f_t(x_1, x_2) \leq 0, t \in T\}$ be a convex system, in \mathbb{R}^2 , where $T = [1, 2[$ and

$$\begin{aligned} f_1(x_1, x_2) &:= |x_1| + [x_2]_+, \\ f_t(x_1, x_2) &:= x_2^2 + (2-t)x_2 - 2t, \quad t \in]1, 2[. \end{aligned}$$

We have $F = \{0\} \times [-2, 0]$ and $T_C = \{1\}$. For all $t \in T \setminus T_C$, $F_t = \left\{ \begin{pmatrix} 0 \\ -2 \end{pmatrix} \right\}$, and (C.1) holds.

On the other hand, for $\bar{x} = 0_2 \in \text{rbd } F$, $T(\bar{x}) = \{1\}$, and $\inf f_1 = 0$. Hence, (C.2) fails.

As in previous examples, this system is LFM:

For $\bar{x} = 0_2$ (it was calculated in Example 11),

$$\partial f_1(\bar{x}) = \{u \in \mathbb{R}^2 \mid -1 \leq u_1 \leq 1, 0 \leq u_2 \leq 1\},$$

then $A(\bar{x}) = \text{cone} \{ \partial f_1(\bar{x}) \} = \mathbb{R} \times [0, +\infty[= D(F, \bar{x})^\circ$.

If $\bar{x} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}$, $T(\bar{x}) = T$, and

$$\begin{aligned} \partial f_1(\bar{x}) &= \{u \in \mathbb{R}^2 \mid -1 \leq u_1 \leq 1, u_2 = 0\}, \\ \partial f_t(\bar{x}) &= \{\nabla f_t(\bar{x})\} = \left\{ \begin{pmatrix} 0 \\ -2-t \end{pmatrix} \right\}, \quad t \in]1, 2[. \end{aligned}$$

(See Example 11 for the calculus of $\partial f_1(\bar{x})$, when $x_2 < 0$.)

Hence, $A(\bar{x}) = \text{cone} \left\{ \bigcup \partial f_t(\bar{x}), t \in T \right\} = \mathbb{R} \times]-\infty, 0] = D(F, \bar{x})^\circ$.

Finally, for $\bar{x} = \begin{pmatrix} 0 \\ x_2 \end{pmatrix}$, $-2 < x_2 < 0$, we get $T(\bar{x}) = \{1\}$, and

$$\partial f_1(\bar{x}) = \{u \in \mathbb{R}^2 \mid -1 \leq u_1 \leq 1, u_2 = 0\},$$

obtaining $A(\bar{x}) = \mathbb{R} \times \{0\} = D(F, \bar{x})^\circ$.

We focus now on the LFM systems, in order to provide sufficient conditions for the fulfilment of the equality (5.2). First, consider the following lemmas.

LEMMA 2. *Let $\sigma = \{f_t(x) \leq 0, t \in T\}$ be a LFM convex system and $\bar{x} \in \text{rbd } F$. Then, there exist $t \in T(\bar{x})$ and $u \in \partial f_t(\bar{x})$ such that, for all $x \in F$, $\langle u, x \rangle \leq \langle u, \bar{x} \rangle$, and $\langle u, z \rangle < \langle u, \bar{x} \rangle$, for some $z \in F$.*

PROOF. Taking account of (4.7) in Remark 6, if $\bar{x} \in \text{rbd } F$ it must be

$$T_L(\bar{x}) \cap (T_L \setminus T_{L,C}) \neq \emptyset.$$

On the other hand, according to [3, (7.11)], we have

$$T_L(\bar{x}) = \{(t, \bar{x}, u) \mid t \in T(\bar{x}) \text{ and } u \in \partial f_t(\bar{x})\}.$$

Hence, there exists $t \in T(\bar{x})$ and $u \in \partial f_t(\bar{x})$ such that $(t, \bar{x}, u) \notin T_{L,C}$, and this implies, since $f_t(\bar{x}) = 0$, that for all $x \in F$, $\langle u, x \rangle \leq \langle u, \bar{x} \rangle$ and $\langle u, z \rangle < \langle u, \bar{x} \rangle$ for some $z \in F$. \square

Let $\sigma = \{f_t(x) \leq 0, t \in T\}$ be a LFM convex system and $\bar{x} \in \text{rbd } F$. We introduce the set

$$U_{\bar{x}} := \{u \in \mathbb{R}^n \mid \langle u, x \rangle \leq \langle u, \bar{x} \rangle, \text{ for all } x \in F; \langle u, z \rangle < \langle u, \bar{x} \rangle, \text{ for some } z \in F\}.$$

Obviously, $U_{\bar{x}} = N(F, \bar{x}) \setminus (F - \bar{x})^\perp$.

LEMMA 3. *Let $\sigma = \{f_t(x) \leq 0, t \in T\}$ be a LFM convex system and $\bar{x} \in \text{rbd } F$. $U_{\bar{x}}$ is a non-empty convex set, such that, if $t \in T(\bar{x})$ verifies $\partial f_t(\bar{x}) \cap U_{\bar{x}} \neq \emptyset$, then $\text{rint } \partial f_t(\bar{x}) \subset U_{\bar{x}}$.*

PROOF. The first statement comes from Lemma 2.

Now, let $t \in T(\bar{x})$ with $\partial f_t(\bar{x}) \cap U_{\bar{x}} \neq \emptyset$. Take $u \in \partial f_t(\bar{x}) \cap U_{\bar{x}}$. If v is any point in $\text{rint } \partial f_t(\bar{x})$ other than u , then, according to Theorem 6.4 in [5], there exists $\mu > 1$ such that $w := (1 - \mu)u + \mu v \in \partial f_t(\bar{x})$. Hence $v = \lambda w + (1 - \lambda)u$, with $\lambda = \frac{1}{\mu}$. Since $\langle w, x \rangle \leq \langle w, \bar{x} \rangle$, for all $x \in F$, it follows that $v \in U_{\bar{x}}$. \square

THEOREM 3. *Let $\sigma = \{f_t(x) \leq 0, t \in T\}$ be a LFM convex system verifying the condition (C.2). Each of the following conditions is sufficient for σ to verify the equality (5.2):*

- (i) $\dim F = n$;
- (ii) $\dim F < n$ and, for all $\bar{x} \in \text{rbd } F$,

$$\left(\bigcup \{ \partial f_t(\text{rint } F), t \in T(\bar{x}) \} \right) \cap U_{\bar{x}} \neq \emptyset. \quad (5.4)$$

PROOF. (i) It was proved in Proposition 7.

(ii) We shall show that, for all $\bar{x} \in \text{rbd } F$, $T(\bar{x}) \cap (T \setminus T_C) \neq \emptyset$.

For $\bar{x} \in \text{rbd } F$, we have $U_{\bar{x}} \neq \emptyset$, by Lemma 3. For each $u \in U_{\bar{x}}$, let us consider the half-space

$$H_u = \{x \in \mathbb{R}^n \mid \langle u, x \rangle \leq \langle u, \bar{x} \rangle\}.$$

We have $F \subset H_u$, but F is not entirely contained in $\text{bd } H_u$. Then, by Corollary 6.5.2 in [5], $\text{rint } F \subset \text{int } H_u$. Let us observe that this condition implies that if $u \in U_{\bar{x}}$, then $\langle u, y \rangle < \langle u, \bar{x} \rangle$, for all $y \in \text{rint } F$.

Since (5.4) holds, there exist $t \in T(\bar{x})$, and $y \in \text{rint } F$ such that $\partial f_t(y) \cap U_{\bar{x}} \neq \emptyset$. Take $v \in \partial f_t(y) \cap U_{\bar{x}}$. Then

$$0 = f_t(\bar{x}) \geq f_t(y) + \langle u, \bar{x} - y \rangle, \text{ and } \langle u, \bar{x} - y \rangle > 0.$$

Hence $f_t(y) < 0$, which implies that $t \notin T_C$. \square

REMARK 8. Now, we can see what fails in the system of Example 11, where $\dim F < n$, and the condition (C.2) holds, but not the equality (5.2): for $\bar{x} = 0_2$, $U_{\bar{x}} = \{u \in \mathbb{R}^2 \mid u_2 > 0\}$, whereas $\bigcup \{\partial f_t(\text{rint } F), t \in T(\bar{x})\} \subset \mathbb{R} \times \{0\}$.

THEOREM 4. Let $\sigma = \{f_t(x) \leq 0, t \in T\}$ be a LFM convex system verifying that, for every $t \in T_C$ and for all $\bar{x} \in \text{rbd } F$,

$$\partial f_t(\bar{x}) \subset (F - \bar{x})^\perp.$$

Then the equality (5.2) holds, as well as the following statements:

(i) For all $\bar{x} \in F$

$$\text{lin } A(\bar{x}) = (F - \bar{x})^\perp = \text{span} \{\cup_{t \in T_C} \cap_{y \in \text{rbd } F} \partial f_t(y)\}.$$

(ii) $\dim F = n - \dim \text{span} \{\cup_{t \in T_C} \cap_{y \in \text{rbd } F} \partial f_t(y)\}.$

PROOF. The first assertion follows again from the fact that $T(\bar{x}) \cap (T \setminus T_C) \neq \emptyset$ if $\bar{x} \in \text{rbd } F$. Actually, by Lemma 2, there exist $t \in T(\bar{x})$ and $u \in \partial f_t(\bar{x})$ such that, for all $x \in F$, $\langle u, x \rangle \leq \langle u, \bar{x} \rangle$ and $\langle u, z \rangle < \langle u, \bar{x} \rangle$ for some $z \in F$. Hence $u \notin (F - \bar{x})^\perp$ and, by hypothesis, $t \notin T_C$.

On the other hand, we know from Theorem 1 (i), that

$$\text{lin } A(\bar{x}) = (F - \bar{x})^\perp = \text{span} \{\cup_{t \in T_C} \cap_{y \in F} \partial f_t(y)\}, \text{ for all } \bar{x} \in F,$$

and

$$\text{span} \{\cup_{t \in T_C} \cap_{y \in F} \partial f_t(y)\} \subset \text{span} \{\cup_{t \in T_C} \cap_{y \in \text{rbd } F} \partial f_t(y)\}.$$

Suppose now that $v \in \partial f_t(y)$, for all $y \in \text{rbd } F$ and certain $t \in T_C$. Then, by hypothesis, $v \in (F - y)^\perp$. Hence, for all $x \in F$ and for all $z \in \mathbb{R}^n$, since $f_t(y) = f_t(x) = 0$, we have

$$f_t(z) \geq \langle v, z - y \rangle = \langle v, z - x \rangle,$$

which implies that $v \in \partial f_t(x)$, for all $x \in F$. Therefore we get (i), and (ii) is a direct consequence of it. \square

The hypothesis stated in Theorems 3 and 4 guarantee the fulfilment of the equality (5.2)

$$\text{rint } F = \{x \in \mathbb{R}^n \mid f_t(x) < 0, t \in T \setminus T_C; f_t(x) = 0, t \in T_C\}.$$

For the case $\dim F = n$, the condition in Theorem 3 is implied by the condition in Theorem 4, since the equality (5.2) is equivalent to (C.2), according to Proposition 7. The following examples show that there are no more implications.

EXAMPLE 13. Let $\sigma = \{f_t(x_1, x_2) \leq 0, t \in T\}$ be a convex system, in \mathbb{R}^2 , where $T = [0, 1] \cup \{2\}$ and

$$\begin{aligned} f_0(x_1, x_2) &:= |x_1| + [x_2]_+, \\ f_t(x_1, x_2) &:= x_1^2 + tx_2, \quad t \in]0, 1], \\ f_2(x_1, x_2) &:= x_1. \end{aligned}$$

The feasible set is $F = \{0\} \times]-\infty, 0]$ and $T_C = \{0, 2\}$. In this case, $T(0_2) = T$ and we have $\inf f_2 < 0$. Let us check that σ is LFM.

For $\bar{x} = 0_2$,

$$\partial f_0(\bar{x}) = \{u \in \mathbb{R}^2 \mid -1 \leq u_1 \leq 1, 0 \leq u_2 \leq 1\}, \quad \partial f_2(\bar{x}) = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\},$$

as we have seen in Example 11.

On the other hand, $\partial f_t(\bar{x}) = \left\{ \begin{pmatrix} 0 \\ t \end{pmatrix} \right\}$, $t \in]0, 1]$. Then

$$A(\bar{x}) = \text{cone} \left\{ \bigcup \partial f_t(\bar{x}), t \in T \right\} = \mathbb{R} \times [0, +\infty[= D(F, \bar{x})^\circ.$$

Now, for $\bar{x} = \begin{pmatrix} 0 \\ x_2 \end{pmatrix}$, $x_2 < 0$, $A(\bar{x}) = D(F, \bar{x})^\circ$, because we have the same active constraints at \bar{x} as in Example 11.

This system verifies condition (ii) in Theorem 3, since for $\bar{x} = 0_2$,

$$U_{\bar{x}} = \{u \in \mathbb{R}^2 \mid u_2 > 0\},$$

and taking any $t \in]0, 1]$, for all $\begin{pmatrix} 0 \\ x_2 \end{pmatrix} \in \text{rint } F$, $\partial f_t(0, x_2) = \left\{ \begin{pmatrix} 0 \\ t \end{pmatrix} \right\} \subset U_{\bar{x}}$. But the hypothesis in Theorem 4, fails, because

$$\partial f_0(\bar{x}) \not\subset (F - \bar{x})^\perp = \mathbb{R} \times \{0\}.$$

EXAMPLE 14. Let $\sigma = \{f_t(x_1, x_2, x_3) \leq 0, t \in T\}$ be a convex system, in \mathbb{R}^3 , where $T = [0, +\infty[$ and

$$\begin{aligned} f_0(x_1, x_2, x_3) &:= \max \left\{ \frac{\sqrt{x_1^2 + x_2^2} - (x_1 + x_2)}{2}, -x_1, -x_2 \right\}, \\ f_t(x_1, x_2, x_3) &:= t|x_3|, \quad t > 0. \end{aligned}$$

The convexity of the function f_0 on \mathbb{R}^3 is a trivial consequence of the convexity of the function

$$g(x_1, x_2, x_3) = \sqrt{x_1^2 + x_2^2}.$$

Actually g is nothing else than the Euclidean norm restricted to the subspace $x_3 = 0$.

We can easily verify that

$$F = \{x \in \mathbb{R}^3 \mid x_1 \geq 0, x_2 \geq 0, x_3 = 0\}$$

and, so, $\dim F = 2$. The next step is to check that σ is LFM and, to this aim, we analyze each $\bar{x} \in F$.

(1) For $\bar{x} = 0_3$, $T(\bar{x}) = T$. It can be seen that, for $t > 0$,

$$\partial f_t(\bar{x}) = \{u \in \mathbb{R}^3 \mid u_1 = u_2 = 0, |u_3| \leq t\}. \quad (5.5)$$

Applying the well-known Valadier formula (see, for instance, [3, Theorem A.17]), Theorem 23.8 in [5], and that

$$\partial g(\bar{x}) = \{u \in \mathbb{R}^3 \mid u_1^2 + u_2^2 \leq 1, u_3 = 0\}$$

(see, for instance, [6, Example VI.3.1]), we have

$$\partial f_0(\bar{x}) = \text{conv} \left(\left\{ \frac{1}{2} \begin{pmatrix} u_1 - 1 \\ u_2 - 1 \\ 0 \end{pmatrix} \mid u_1^2 + u_2^2 \leq 1 \right\} \cup \left\{ \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} \right\} \right). \quad (5.6)$$

From (5.5) and (5.6), we get

$$\begin{aligned} A(\bar{x}) &= \text{cone} \left\{ \partial f_0(\bar{x}) \cup \left(\bigcup_{t>0} \partial f_t(\bar{x}) \right) \right\} \\ &=]-\infty, 0] \times]-\infty, 0] \times \mathbb{R} \\ &= D(F, \bar{x})^\circ. \end{aligned}$$

(2) For $\bar{x} = \begin{pmatrix} x_1 \\ 0 \\ 0 \end{pmatrix}$, with $x_1 > 0$, again $T(\bar{x}) = T$.

For $t > 0$, (5.5) holds, but now g is differentiable and in this case, $\partial g(\bar{x}) = \{\nabla g(\bar{x})\} = \left\{ \begin{pmatrix} 0 \\ -\frac{1}{2} \\ 0 \end{pmatrix} \right\}$, hence

$$\partial f_0(\bar{x}) = \text{conv} \left\{ \begin{pmatrix} 0 \\ -\frac{1}{2} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} \right\}. \quad (5.7)$$

From (5.5) and (5.7) we obtain

$$A(\bar{x}) = \{0\} \times]-\infty, 0] \times \mathbb{R} = D(F, \bar{x})^\circ.$$

(3) For $\bar{x} = \begin{pmatrix} 0 \\ x_2 \\ 0 \end{pmatrix}$, $x_2 > 0$, use a symmetric reasoning to the one followed in (2).

(4) For $\bar{x} = \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix}$, $x_1 > 0$ and $x_2 > 0$, $T(\bar{x}) =]0, +\infty[$, and (5.5) yields

$$A(\bar{x}) = \{0_2\} \times \mathbb{R} = D(F, \bar{x})^\circ.$$

Therefore, σ is a convex LFM system.

The assumptions in Theorem 4 are satisfied since, for every $t \in T_C =]0, +\infty[$, and for all $\bar{x} \in \text{rbd } F$, we have

$$\partial f_t(\bar{x}) = \{u \in \mathbb{R}^3 \mid u_1 = u_2 = 0, |u_3| \leq t\} \subset (F - \bar{x})^\perp.$$

However, the assumptions in Theorem 3 are not fulfilled. Actually, if $\bar{x} = \begin{pmatrix} x_1 \\ 0 \\ 0 \end{pmatrix}$, with $x_1 > 0$, we have $T(\bar{x}) = T$ and

$$U_{\bar{x}} = \{u \in \mathbb{R}^3 \mid u_1 = 0, u_2 < 0\}.$$

If $x = \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} \in \text{rint } F$, we observe that $\partial f_t(x) \cap U_{\bar{x}} = \emptyset$, for all $t > 0$. Moreover

$$\partial f_0(x) = \{\nabla f_0(x)\} = \left\{ \frac{1}{2} (x_1^2 + x_2^2)^{-\frac{1}{2}} \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

Since $x_2 \neq 0$, we have

$$x_1 (x_1^2 + x_2^2)^{-\frac{1}{2}} - 1 < 0,$$

and we conclude that $\partial f_0(x) \cap U_{\bar{x}} = \emptyset$.

EXAMPLE 15. Let us consider the convex system $\sigma = \{f_t(x) \leq 0, t \in T\}$, in \mathbb{R} , where $T = [1, 2]$ and

$$\begin{aligned} f_1(x) &:= x^2 - 1, \\ f_t(x) &:= \begin{cases} -tx - t, & \text{if } x \leq 1, \\ 0, & \text{if } -1 < x < 1, \\ tx - t, & \text{if } x \geq 1. \end{cases} \quad , \text{ for } 1 < t \leq 2. \end{aligned}$$

The feasible set is $F = [-1, 1]$, $T_C =]1, 2]$ and $F_1 = \{-1, 1\}$. The system σ is LFM and, for $\bar{x}_1 = -1, \bar{x}_2 = 1$, we can take $t = 1$, which verifies $\inf f_1 < 0$. Since $\dim F = 1$, we are in case (i) of Theorem 3. But conditions of Theorem 4 are not held for this system: take $\bar{x} = -1$ and $2 \in T_C$. Then $-2 \in \partial f_2(\bar{x})$, but $(F - \bar{x})^\perp = \{0\}$.

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